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Fundamental weights, permutation weights and Weyl character formula

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Abstract. For a finite Lie algebra G_N of rank N, the Weyl orbits $W(\Lambda^{++})$ of strictly dominant weights Λ^{++} contain $\dim W(G_N)$ number of weights, where $\dim W(G_N)$ is the dimension of its Weyl group $W(G_N)$. For any $W(\Lambda^{++})$, there is a very peculiar subset $\varrho(\Lambda^{++})$ for which we always have

$$\dim \varrho(\Lambda^{++}) = \dim W(G_N) / \dim W(A_{N-1}).$$

For any dominant weight Λ^{+}, the elements of $\varrho(\Lambda^{+})$ are called *permutation weights*.

It is shown that there is a one-to-one correspondence between the elements of $\varrho(\Lambda^{++})$ and $\varrho(\rho)$ where ρ is the Weyl vector of G_N. The concept of the signature factor which enters the Weyl character formula can be relaxed in such a way that signatures are preserved under this one-to-one correspondence in the sense that corresponding permutation weights have the same signature. Once the permutation weights and their signatures are specified for a dominant Λ^{+}, calculation of the character $\mathrm{Ch}\,R(\Lambda^{+})$ for the irreducible representation $R(\Lambda^{+})$ will then be provided by A_N multiplicity rules governing the generalized Schur functions. The main idea is again to express everything in terms of the so-called *fundamental weights* with which we obtain a quite relevant specialization in applications of the Weyl character formula. To provide simplifications in practical calculations, a reduction formula governing the classical Schur functions is also given. As the most suitable example, E_6, which requires a sum over 51 840 Weyl group elements, is studied explicitly. This will be instructive also for an explicit application of A_5 multiplicity rules.

As a result, it will be seen that the Weyl or Weyl–Kac character formulae find explicit applications no matter how large the rank of the underlying algebra.

1. Introduction

It is well known that summations over Weyl groups of Lie algebras enter many areas of physics as well as mathematics. They are at the heart of all character calculations for finite [1] and also affine [2] Lie algebras and hence are of great importance in calculations of weight multiplicities [3] or in decompositions [4] of tensor products of irreducible representations. In high-energy physics, it is known that calculations of fusion coefficients [5] or S-matrices which appear in modular transformations [6] of affine characters are directly related to summations over Weyl groups. This, however, is not an easy task, except for a few cases which correspond to some Lie algebras of low rank. Let us emphasize, for instance, that the summations are over 51 840, 2 903 040 and 696 729 600 Weyl group elements for E_6, E_7 and E_8 Lie algebras, respectively. It is therefore worthwhile to study the problem more closely.

In a previous work [7] we showed that in applications of the Weyl character formula for A_N Lie algebras the sums over Weyl groups can be represented by permutations. This is

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in essence in line with the fact that A_N Weyl groups are already the permutation groups of $(N+1)$ objects. It is interesting to note, however, that this can be seen only when one uses some properly chosen set of weights which we call *fundamental weights*. We also showed that the signatures of the Weyl reflections can then be given precisely. One could therefore expect there to be a way of extending this procedure to any other finite Lie algebra $\mathbf{G}_N$ in view of the fact that it always has an A_{N-1} subalgebra.

For this, let us recall that for any dominant weight Λ^+ of $\mathbf{G}_N$ one has a strictly dominant weight $\Lambda^{++} \equiv \Lambda^+ + \rho$, where ρ is the Weyl vector of $\mathbf{G}_N$. The character $\mathrm{Ch}\, R(\Lambda^+)$ of the corresponding irreducible representation $R(\Lambda^+)$ is then given by

$$\mathrm{Ch}\, R(\Lambda^+) = \frac{A(\Lambda^{++})}{A(\rho)} \tag{1.1}$$

where

$$A(\mu) \equiv \sum_{\omega} \epsilon(\omega)\, e^{\omega(\mu)} \tag{1.2}$$

can be defined for any weight μ. The sum here is over the Weyl group $W(G_N)$ and $\epsilon(\omega)$ is the signature of the Weyl reflection ω. The main emphasis now is on the fact that, for any strictly positive dominant weight Λ^{++}, the number of elements of Weyl orbit $W(\Lambda^{++})$ is always equal to the dimension of the corresponding Weyl group. Hence this allows us to re-write (1.2) in the form

$$A(\Lambda^{++}) \equiv \sum_{\mu \in W(\Lambda^{++})} \epsilon(\mu) e^{\mu} \tag{1.3}$$

where $W(\Lambda^{++})$ is the corresponding Weyl orbit. One must immediately note here that the concept of signature encountered in (1.2) is conveniently relaxed in (1.3) in such a way that we introduce a signature $\epsilon(\mu)$ for each and every weight μ within the Weyl orbit $W(\Lambda^{++})$. It will be seen in what follows that (1.3) is a quite relevant form of (1.2) if one aims to use it in the Weyl character formula (1.1). To this end, the concept of *permutation weight* is of central importance.

2. Permutation weights

It is known that the branching rules $\mathbf{G}_N \rightarrow A_{N-1}$ give us irreducible A_{N-1} representations which participate in the decomposition of an irreducible representation of G_N. Instead, here we want to do the same for Weyl orbits rather than representations. For this, the following definition seems to be useful.

A Weyl orbit $W(\Lambda^+)$ always includes a subset $\varrho(\Lambda^+)$ of weights having the form

$$\sum_{i=1}^{N-1} k_i\, \lambda_i - k\, \lambda_N \qquad k_i \in \mathbb{Z}^+ \quad k \in \mathbb{Z} \tag{2.1}$$

where $\mathbb{Z}$ ($\mathbb{Z}^+$) is the set of integers (positive integers). The elements of $\varrho(\Lambda^+)$ are called the *permutation weights* of Λ^+.

The λ_I and α_I $(I = 1, 2, \ldots, N)$ are respectively the fundamental dominant weights and the simple roots of G_N. For details of Lie algebra technology we refer to the excellent book of Humphreys [8]. As will be seen from the permutational lemma given in our previous work [9], the Weyl orbits of A_N Lie algebras are stable under permutations and hence this allows us to determine the complete weight structure of an A_N Weyl orbit. The permutation weights will give us the same possibility, but for any finite Lie algebra G_N other than A_N Lie algebras. We will therefore show now an explicit way of obtaining all permutation weights of a Weyl orbit $W(\Lambda^+)$ of G_N.

Let us first emphasize by definition that the sum of two permutation weights is again a permutation weight. Let $\varrho(\lambda)$ and $\varrho(\lambda')$ be the sets of permutation weights for λ and λ'. It is then clear that

$$\varrho(\lambda + \lambda') \subset \varrho(\lambda) \cup \varrho(\lambda') \tag{2.2}$$

and for any element $\mu \in \varrho(\lambda) \cup \varrho(\lambda')$ one can also state $\mu \in \varrho(\lambda + \lambda')$ on condition that

$$(\mu, \mu) = (\lambda + \lambda', \lambda + \lambda') \tag{2.3}$$

where $(\cdot, \cdot)$ is the scalar product which can be introduced on the weight lattice of $\mathbf{G}_N$. It is therefore sufficient to know the $\varrho(\lambda_I)$ $(I = 1, 2, \dots, N)$ in order to obtain the set $\varrho(\Lambda^+)$ of permutation weights for any dominant weight Λ^+ which is known to be expressed by

$$\Lambda^+ = \sum_{I=1}^{N} k_I \lambda_I \qquad k_I \in \mathbb{Z}^+.$$

We find it convenient here to exemplify our work in the Lie algebra of E_6 with the following Coxeter–Dynkin diagram:

6

1 2 3 4 5

The permutation weight subsets of its fundamental Weyl orbits will then be given by

$$\begin{aligned}
&\varrho(\lambda_1) \equiv \{\lambda_1\ ,\ \lambda_1 - \lambda_6\ ,\ \lambda_4 - \lambda_6\}\\
&\varrho(\lambda_2) \equiv \{\lambda_2\ ,\ \lambda_2 - 2\lambda_6\ ,\ \lambda_3 + \lambda_5 - 2\lambda_6\ ,\ \lambda_1 + \lambda_4 - \lambda_6\ ,\ \lambda_1 + \lambda_4 - 2\lambda_6\ ,\ 2\lambda_1 - \lambda_6\}\\
&\varrho(\lambda_3) \equiv \{\lambda_3\ ,\ \lambda_1 + \lambda_3 + \lambda_5 - 3\lambda_6\ ,\ \lambda_1 + \lambda_3 + \lambda_5 - 2\lambda_6\ ,\ \lambda_2 + 2\lambda_5 - 2\lambda_6\ ,\\
&\qquad\qquad \lambda_2 + \lambda_4 - 3\lambda_6\ ,\ \lambda_2 + \lambda_4 - \lambda_6\ ,\ \lambda_3 - 3\lambda_6\ ,\ 2\lambda_3 - 3\lambda_6\ ,\ 2\lambda_1 + \lambda_4 - 2\lambda_6\}\\
&\varrho(\lambda_4) \equiv \{\lambda_4\ ,\ \lambda_4 - 2\lambda_6\ ,\ \lambda_1 + \lambda_3 - 2\lambda_6\ ,\ \lambda_2 + \lambda_5 - \lambda_6\ ,\ \lambda_2 + \lambda_5 - 2\lambda_6\ ,\ 2\lambda_5 - \lambda_6\}\\
&\varrho(\lambda_5) \equiv \{\lambda_5\ ,\ \lambda_2 - \lambda_6\ ,\ \lambda_5 - \lambda_6\}\\
&\varrho(\lambda_6) \equiv \{\lambda_6\ ,\ -\lambda_6\ ,\ \lambda_1 + \lambda_5 - \lambda_6\ ,\ \lambda_3 - \lambda_6\ ,\ \lambda_3 - 2\lambda_6\}.
\end{aligned} \tag{2.4}$$

In the notation of $(k_1, k_2, k_3, k_4, k_5, k_6)$ for $\sum_{I=1}^{6} k_I\ \lambda_I$, half of the 72 elements of $\varrho(\rho)$ can now be chosen, by direct use of (2.3), from the elements of $\sum_{I=1}^{6} \varrho(\lambda_I)$ as follows:

$\rho(1) = (1, 1, 1, 1, 1, 1)$	$\rho(13) = (3, 2, 2, 1, 2, -6)$	$\rho(25) = (3, 1, 3, 2, 1, -8)$
$\rho(2) = (1, 1, 2, 1, 1, -1)$	$\rho(14) = (2, 1, 2, 2, 3, -6)$	$\rho(26) = (6, 1, 1, 2, 1, -7)$
$\rho(3) = (1, 2, 1, 2, 1, -2)$	$\rho(15) = (1, 3, 1, 3, 1, -6)$	$\rho(27) = (1, 2, 1, 1, 6, -7)$
$\rho(4) = (1, 3, 1, 1, 2, -3)$	$\rho(16) = (4, 2, 1, 1, 3, -6)$	$\rho(28) = (2, 2, 2, 1, 4, -8)$
$\rho(5) = (2, 1, 1, 3, 1, -3)$	$\rho(17) = (3, 1, 1, 2, 4, -6)$	$\rho(29) = (4, 1, 2, 2, 2, -8)$
$\rho(6) = (2, 2, 1, 2, 2, -4)$	$\rho(18) = (4, 1, 3, 1, 1, -7)$	$\rho(30) = (2, 1, 4, 1, 2, -9)$
$\rho(7) = (1, 4, 1, 1, 1, -4)$	$\rho(19) = (1, 1, 3, 1, 4, -7)$	$\rho(31) = (7, 1, 1, 1, 1, -7)$
$\rho(8) = (1, 1, 1, 4, 1, -4)$	$\rho(20) = (2, 2, 2, 2, 2, -7)$	$\rho(32) = (1, 1, 1, 1, 7, -7)$
$\rho(9) = (3, 1, 2, 1, 3, -5)$	$\rho(21) = (5, 1, 2, 1, 2, -7)$	$\rho(33) = (1, 3, 1, 1, 5, -8)$
$\rho(10) = (2, 3, 1, 2, 1, -5)$	$\rho(22) = (2, 1, 2, 1, 5, -7)$	$\rho(34) = (5, 1, 1, 3, 1, -8)$
$\rho(11) = (1, 2, 1, 3, 2, -5)$	$\rho(23) = (3, 2, 1, 2, 3, -7)$	$\rho(35) = (3, 1, 3, 1, 3, -9)$
$\rho(12) = (4, 1, 1, 1, 4, -5)$	$\rho(24) = (1, 2, 3, 1, 3, -8)$	$\rho(36) = (1, 1, 5, 1, 1, -10).$

(2.5)

3. Explicit construction of Weyl orbits

It is known that the complete set of weights of a Weyl orbit is obtained from the fact that the Weyl orbits are by definition stable under Weyl reflections. Instead, we want to construct Weyl orbits here solely by knowing their permutation weights. As above, let the λ_I be the fundamental dominant weights of $\mathbf{G}_N$, and let the σ_i be those of its A_{N-1} subalgebra.

The existence of such a subalgebra can always be shown explicitly by taking

$$\sigma_i = \lambda_i - n_i \lambda_N \tag{3.1}$$

where the n_i are some specified rational numbers. Let us recall from our previous articles [7, 9] that the fundamental weights μ_I ($I = 1, 2, \ldots, N$) for the A_{N-1} subalgebra are defined by

$$\sigma_i \equiv \mu_1 + \mu_2 + \cdots + \mu_i \qquad i = 1, 2, \ldots, N-1 \tag{3.2}$$

together with the condition that

$$\mu_1 + \mu_2 + \cdots + \mu_N \equiv 0 \tag{3.3}$$

and also

$$(\mu_I, \lambda_N) \equiv 0. \tag{3.4}$$

The permutational lemma then states for an A_{N-1} dominant weight

$$\sigma^+ = s_1\mu_1 + s_2\mu_2 + \cdots + s_N\mu_N \qquad s_1 \geqslant s_2 \geqslant \cdots \geqslant s_N \geqslant 0 \tag{3.5}$$

that its Weyl orbits $W(\sigma^+)$ are obtained as

$$W(\sigma^+) = \{s_1\mu_{I_1} + s_2\mu_{I_2} + \cdots + s_N\mu_{I_N}\} \tag{3.6}$$

by permuting the fundamental weights μ_I. Note here that no two of the indices $I_1, I_2, \ldots, I_N$ ($= 1, 2, \ldots, N$) take the same value. This is also true for all permutation weights because, for $\lambda_N \to 0$, they turn out to be A_{N-1} dominant weights. We then obtain an extension of the permutational lemma for any finite Lie algebra other than the A_N Lie algebras.

An example will again be helpful here. Let us consider the $E_6 \to A_5$ decomposition which is specified by

$$\begin{aligned} &\sigma_1 = \lambda_1 - \tfrac{1}{2}\lambda_6 \qquad \sigma_2 = \lambda_2 - \tfrac{2}{2}\lambda_6 \qquad \sigma_3 = \lambda_3 - \tfrac{3}{2}\lambda_6 \\ &\sigma_4 = \lambda_4 - \tfrac{2}{2}\lambda_6 \qquad \sigma_5 = \lambda_5 - \tfrac{1}{2}\lambda_6 \end{aligned} \tag{3.7}$$

where the λ_I ($I = 1, 2, \ldots, 6$) are the E_6 fundamental dominant weights, while the σ_i ($i = 1, 2, \ldots, 5$) are those of A_5. The influence of the A_5 permutational lemma for the E_6 Weyl orbits can be illustrated, in view of (2.4), in the following example:

$$W(\lambda_1) = \{W(\sigma_1) + \tfrac{1}{2}\Omega\ ,\ W(\sigma_1) - \tfrac{1}{2}\Omega\ ,\ W(\sigma_4)\} \tag{3.8}$$

where, for A_5 Weyl orbits, we know that

$$W(\sigma_1) = \{\mu_{I_1}\}\ ,\ W(\sigma_4) = \{\mu_{I_1} + \mu_{I_2} + \mu_{I_3} + \mu_{I_4}\} \qquad I_1 \geqslant I_2 \geqslant I_3 \geqslant I_4 = 1, 2, \ldots, 6.$$

In (3.8) one keeps the notation $\Omega \equiv \lambda_6$, for which we know that $(\Omega, \mu_I) = 0$. It is, in fact, simply an example of the *branching rule of Weyl orbits* which is at the heart of our definition of permutation weights. The branching rules for the remaining fundamental E_6 Weyl orbits $W(\lambda_i)$ for $i = 2, 3, \ldots, 6$ can be similarly obtained from the permutation weights given in (2.4).

What we want to emphasize here is mainly that the 72 permutation weights of $W(\rho)$ of E_6 will be given by

$$\varrho(\rho) = (\sigma(k)^{++} + r(k)\ \Omega\ ,\ \sigma(k)^{++} - r(k)\ \Omega) \tag{3.9}$$

where the Ω extension parameters $r(k)$ are some *positive* rational numbers. The 36 strictly dominant weights $\sigma(k)^{++}$ and their parameters $r(k)$ can respectively be determined from (2.5) ($k = 1, 2, \ldots, 36$).

4. Applications of the Weyl formula and a lemma

Since we have in mind to perform the summations over the Weyl groups explicitly, the decomposition (3.9) gives us the possibility for E_6 of calculating $A(\rho)$ with the aid of (1.3). Let us recall from [7] that for any one of the A_5 dominant weights

$$\sigma^{++}(k) \equiv \sigma^{+}(k) + \rho_\sigma$$

in the list (3.9), one has

$$A(\sigma^{++}(k) + r(k)\ \Omega) = A(\rho_\sigma)\ S(\sigma^{+}(k))\ u^{r(k)} \tag{4.1}$$

where ρ_σ is the Weyl vector of A_5. In the specialization

$$e^{\Omega} \equiv u \qquad e^{\mu_I} \equiv u_I \qquad I = 1, 2, \ldots, 6 \tag{4.2}$$

of formal exponentials, we know that $S(\sigma^{+}(k))$ is a generalized Schur function [7, 10] which can be reduced via the A_5 multiplicity rules to a polynomial expression in terms of five indeterminates x_i $(i = 1, 2, \ldots, 5)$ which are defined by

$$u_1^M + u_2^M + u_3^M + u_4^M + u_5^M + u_6^M \equiv M\ x_M \qquad M = 1, 2, \ldots . \tag{4.3}$$

Note here as a result of (3.3) that six indeterminates u_I are constrained by

$$\prod_{I=1}^{6} u_I \equiv 1 \tag{4.4}$$

and hence one can immediately see from definitions (4.3) that, for $M > 5$, all indeterminates x_M depend nonlinearly on the first five indeterminates x_i $(i = 1, 2, \ldots, 5)$. This will also give rise to some reduction rules governing classical Schur functions [7] which are defined by

$$S(M\ \lambda_1) \equiv S_M(x_1, x_2, \ldots, x_5) \qquad M = 1, 2, \ldots, 5, 6, \ldots \tag{4.5}$$

where the $S_M(x_1, x_2, \ldots, x_5)$ are some polynomials which can be obtained for $M = 1, 2, \ldots, 5$ directly. For $M > 5$, however, one must take into account the above-mentioned nonlinear relations between the indeterminates x_M. Practical calculations could become complicated in general for A_N multiplicity rules. For this, we find it convenient to give some clarifying details here. It will be seen in fact that these nonlinear relations governing the indeterminates x_M for $M > N$ result in the following reduction rules between the polynomials $S_M(x_1, x_2, \ldots, x_N) \equiv S_M(N)$ which correspond to the classical Schur functions as in (4.5):

$$S_M(N) = (-1)^N\ S_{M-N-1}(N) - \sum_{i=1}^{N} S_i^*(N)\ S_{M-i}(N) \qquad M > N \tag{4.6}$$

where $S_M^*(N)$ is obtained from $S_M(N)$ under the replacements $x_i \to -x_i$. It will be seen that the reduction rules given in (4.6) prove extremely useful in applications of A_N multiplicity rules, especially for higher values of the rank N.

Another important point to note here is to give a precise definition of the signatures for the 72 permutation weights in the decomposition (3.9). The arrangement in (3.9) is in such a way that

$$\begin{aligned} &\epsilon(\sigma^{++}(k) + r(k)\ \Omega) \equiv +1 \\ &\epsilon(\sigma^{++}(k) - r(k)\ \Omega) \equiv -1 \end{aligned} \tag{4.7}$$

for $k = 1, 2, \ldots, 36$. The miraculous factorization (4.1) of the Weyl formula comes out only with the aid of such a choice.

It can thus be seen that the decomposition (3.9) of $\varrho(\rho)$ allows us to calculate $A(\rho)$ but nothing is said about any other $A(\Lambda^{++})$ which we need in the calculation of the character

Ch $R(\Lambda^+)$. For this, a lemma which assures one-to-one correspondence between the 72 elements of (3.9) and those of any other $\varrho(\Lambda^{++})$ would be of great help. In view of condition (2.3), there is a one-to-one correspondence which maps any element of $\varrho(\rho)$ to one and only one element of $\varrho(\Lambda^{++})$ in such a way that their signatures are preserved. The generalization leads us to the following lemma.

Lemma. *Let, for any dominant weight Λ^+, $\varrho(\Lambda^+)$ be the subset of its permutation weights and also*

$$\varrho(\rho) \equiv \{\rho(k)\} \qquad \varrho(\Lambda^{++}) \equiv \{\Lambda(k)\} \tag{4.8}$$

for any Lie algebra G_N with the Weyl vector ρ. Then, in view of condition (2.3), for

$$k = 1, 2, \ldots, \frac{\dim W(G_N)}{\dim W(A_{N-1})}$$

and $\mu \in \varrho(\Lambda^+)$ there is a one-to-one correspondence Ξ which provides

$$\Xi \ : \ \rho(k) + \mu \rightarrow \Lambda(k) \tag{4.9}$$

in such a way that

$$\epsilon(\Xi(\rho(k))) \equiv \epsilon(\rho(k)). \tag{4.10}$$

Note here that we always have

$$\dim \varrho(\Lambda^{++}) \geqslant \dim \varrho(\Lambda^+)$$

and for each and every value of k there is one and only one $\mu \in \varrho(\Lambda^+)$.

In conclusion, we can say that the decomposition (2.5) makes any explicit summation over the 51 840 elements of the E_6 Weyl group possible and hence completely solves the problem for the E_6 Lie algebra. One must add, however, that the related definitions must be made precisely case by case for any other Lie algebra. For all the chains B_N, C_N, D_N, the exceptional Lie algebras G_2, F_4 and even for E_7 the method presented above is tractable as we will show in a subsequent paper. The same could also be true for E_8 but again one must note that we have $\dim \varrho(\rho) = 17\,280$ for E_8. We finally remark that a similar analysis can be presented in the framework of the A_8 subalgebra of E_8, this making the problem more tractable by reducing the number of permutation weights down to 1920. To the knowledge of the authors, this is quite convenient for handling any problem which requires summations over the 696 729 600 elements of the E_8 Weyl group in an explicit manner, and hence it would be worth studying in another publication.

Last but not least, let us add that all these calculations can be performed by the aid of very simple computer programs, say, in the language *Mathematica*™ [11].

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